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SHORT NOTES

Strain determination from three known stretches—an exact solution

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Abstract—In a domain of finite homogeneous deformation, the strain ellipse is uniquely specified by three linearly independent stretch measurements. An exact determination of the principal reciprocal quadratic stretches and principal directions is obtained by solving three simultaneous equations. The equations derived have important applications to the problem of constructing the strain ellipsoid from sectional data and to finite-element interpolation.

INTRODUCTION

DETERMINATION of the strain ellipse from three measured stretches is a classic problem in geological strain analysis. Ramsay (1967, pp. 80–81) presented a graphical approach using the Mohr circle construction. This solution was also adopted by Ramsay & Huber (1983, pp. 91–104), whilst Sanderson (1977) and Ragan (1987) introduced alternative procedures. The existence of an exact solution using three simultaneous quadratic equations has been alluded to by previous workers, but Ragan's method of matrix inversion and eigenvector determination is the only direct treatment available. The derivation in this paper yields the elusive solution to the three simultaneous stretch equations; it differs from Ragan's approach in that one-line expressions for principal directions and reciprocal quadratic stretches are obtained. As in all previous treatments, the analysis is two-dimensional and strain homogeneity is assumed.

DERIVATION

Following standard notation, λ' denotes the reciprocal quadratic stretch of a line and ϕ its orientation relative to an arbitrary zero direction in the deformed state. Three longitudinal strain markers are distinguished by subscripts a, b, c and the principal directions in the deformed state are labelled 1 and 2. From the equation of an ellipse oriented oblique to its reference frame (Fig. 1)

$$\frac{\lambda_a \cos^2(\phi_a - \phi_1)}{\lambda_1} + \frac{\lambda_a \sin^2(\phi_a - \phi_1)}{\lambda_2} = 1, \quad (1)$$

where λ_a denotes the squared length of an arbitrary radius at an angle ϕ_a to the zero direction. Rearranging, writing λ' for $1/\lambda$, and repeating for subscripts b and c,

$$\lambda'_a = \lambda'_1 \cos^2(\phi_a - \phi_1) + \lambda'_2 \sin^2(\phi_a - \phi_1) \quad (2)$$

$$\lambda'_b = \lambda'_1 \cos^2(\phi_b - \phi_1) + \lambda'_2 \sin^2(\phi_b - \phi_1) \quad (3)$$

$$\lambda'_c = \lambda'_1 \cos^2(\phi_c - \phi_1) + \lambda'_2 \sin^2(\phi_c - \phi_1) \quad (4)$$

(compare Ramsay 1967, equation 3-31). Dividing equation (2) by $\sin^2(\phi_a - \phi_1)$ and equation (3) by $\sin^2(\phi_b - \phi_1)$,

$$\frac{\lambda'_a}{\sin^2(\phi_a - \phi_1)} = \frac{\lambda'_1}{\tan^2(\phi_a - \phi_1)} + \lambda'_2 \quad (5)$$

$$\frac{\lambda'_b}{\sin^2(\phi_b - \phi_1)} = \frac{\lambda'_1}{\tan^2(\phi_b - \phi_1)} + \lambda'_2, \quad (6)$$

then subtracting equation (6) from equation (5) to eliminate λ'_2,

$$\frac{\lambda'_a}{\sin^2(\phi_a - \phi_1)} - \frac{\lambda'_1}{\tan^2(\phi_a - \phi_1)} = \frac{\lambda'_b}{\sin^2(\phi_b - \phi_1)} - \frac{\lambda_1{}'}{\tan^2(\phi_b - \phi_1)}. \quad (7)$$

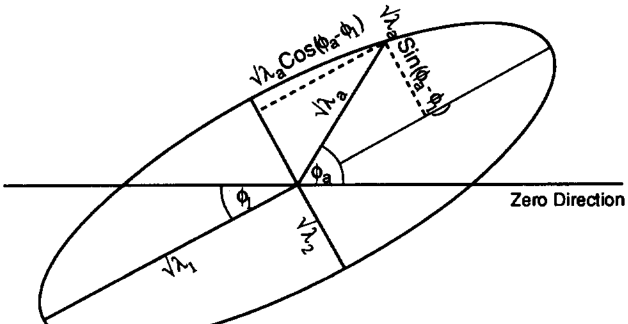

Fig. 1. Strain ellipse with strain marker *a*. Simple trigonometry yields equation (1).

Similarly, λ_1' may be eliminated to yield

$$\frac{\lambda_a'}{\cos^2(\phi_a - \phi_1)} - \lambda_2' \tan^2(\phi_a - \phi_1) = \frac{\lambda_b'}{\cos^2(\phi_b - \phi_1)} - \lambda_2' \tan^2(\phi_b - \phi_1). \quad (8)$$

Equations (7) and (8) may be rearranged thus,

$$\lambda_1' = \frac{\dfrac{\lambda_a'}{\sin^2(\phi_a - \phi_1)} - \dfrac{\lambda_b'}{\sin^2(\phi_b - \phi_1)}}{\dfrac{1}{\tan^2(\phi_a - \phi_1)} - \dfrac{1}{\tan^2(\phi_b - \phi_1)}} \quad (9)$$

$$\lambda_2' = \frac{\dfrac{\lambda_a'}{\cos^2(\phi_a - \phi_1)} - \dfrac{\lambda_b'}{\cos^2(\phi_b - \phi_1)}}{\tan^2(\phi_a - \phi_1) - \tan^2(\phi_b - \phi_1)}. \quad (10)$$

These are expressions for the principal reciprocal quadratic stretches in terms of two arbitrary longitudinal strain markers a and b, and the maximum principal stretch direction ϕ_1. In order to eliminate the latter, the third strain marker, c, must be employed.

From the well known properties of Mohr circle constructions (e.g. Ramsay 1967, p. 69), equations (2), (3) and (4) may be rewritten thus,

$$\lambda_a' = \frac{\lambda_1' + \lambda_2'}{2} + \frac{\lambda_1' - \lambda_2'}{2} \cos 2(\phi_a - \phi_1) \quad (11)$$

$$\lambda_b' = \frac{\lambda_1' + \lambda_2'}{2} + \frac{\lambda_1' - \lambda_2'}{2} \cos 2(\phi_b - \phi_1) \quad (12)$$

$$\lambda_c' = \frac{\lambda_1' + \lambda_2'}{2} + \frac{\lambda_1' - \lambda_2'}{2} \cos 2(\phi_c - \phi_1). \quad (13)$$

Subtracting equation (12) from equation (11) and equation (13) from equation (12),

$$\lambda_a' - \lambda_b' = \frac{\lambda_1' - \lambda_2'}{2} [\cos 2(\phi_a - \phi_1) - \cos 2(\phi_b - \phi_1)] \quad (14)$$

$$\lambda_b' - \lambda_c' = \frac{\lambda_1' - \lambda_2'}{2} [\cos 2(\phi_b - \phi_1) - \cos 2(\phi_c - \phi_1)]. \quad (15)$$

Eliminating the common factor, $(\lambda_1' - \lambda_2')/2$, by cross-division,

$$\frac{\lambda_a' - \lambda_b'}{\cos 2(\phi_a - \phi_1) - \cos 2(\phi_b - \phi_1)} = \frac{\lambda_b' - \lambda_c'}{\cos 2(\phi_b - \phi_1) - \cos 2(\phi_c - \phi_1)} \quad (16)$$

or, in other words,

$$\begin{aligned} &\lambda_a'[\cos 2(\phi_b - \phi_1) - \cos 2(\phi_c - \phi_1)] \\ &+ \lambda_b'[\cos 2(\phi_c - \phi_1) - \cos 2(\phi_a - \phi_1)] \\ &+ \lambda_c'[\cos 2(\phi_a - \phi_1) - \cos 2(\phi_b - \phi_1)] = 0. \quad (17) \end{aligned}$$

Regrouping the factors in equation (17),

$$\begin{aligned} &(\lambda_a' - \lambda_b') \cos 2(\phi_c - \phi_1) + (\lambda_b' - \lambda_c') \cos 2(\phi_a - \phi_1) \\ &+ (\lambda_c' - \lambda_a') \cos 2(\phi_b - \phi_1) = 0. \quad (18) \end{aligned}$$

Using the cosine rule, $\cos(A - B) = \cos A . \cos B + \sin A . \sin B$, equation (18) becomes

$$\begin{aligned} &[(\lambda_a' - \lambda_b') \cos 2\phi_c + (\lambda_b' - \lambda_c') \cos 2\phi_a \\ &+ (\lambda_c' - \lambda_a') \cos 2\phi_b] \cos 2\phi_1 \\ &+ [(\lambda_a' - \lambda_b') \sin 2\phi_c + (\lambda_b' - \lambda_c') \sin 2\phi_a \\ &+ (\lambda_c' - \lambda_a') \sin 2\phi_b] \sin 2\phi_1 = 0. \quad (19) \end{aligned}$$

To solve for the unknown principal direction, divide equation (19) by $\cos 2\phi_1$, and then by the coefficient of $\sin 2\phi_1$,

$$\tan 2\phi_1 = -\left[\frac{(\lambda_a' - \lambda_b') \cos 2\phi_c + (\lambda_b' - \lambda_c') \cos 2\phi_a + (\lambda_c' - \lambda_a') \cos 2\phi_b}{(\lambda_a' - \lambda_b') \sin 2\phi_c + (\lambda_b' - \lambda_c') \sin 2\phi_a + (\lambda_c' - \lambda_a') \sin 2\phi_b}\right]$$

or

$$\phi_1 = -\frac{1}{2} \arctan \left[\frac{(\lambda_a' - \lambda_b') \cos 2\phi_c + (\lambda_b' - \lambda_c') \cos 2\phi_a + (\lambda_c' - \lambda_a') \cos 2\phi_b}{(\lambda_a' - \lambda_b') \sin 2\phi_c + (\lambda_b' - \lambda_c') \sin 2\phi_a + (\lambda_c' - \lambda_a') \sin 2\phi_b}\right]. \quad (20)$$

Equation (20) may be written more succinctly using summations ranging over a permutation of the subscripts.

$$\phi_1 = -\frac{1}{2} \arctan \left[\frac{\sum_{a,b,c} (\lambda_a' - \lambda_b') \cos 2\phi_c}{\sum_{a,b,c} (\lambda_a' - \lambda_b') \sin 2\phi_c}\right]. \quad (21)$$

Equations (9) and (10) may also be tidied by writing $\phi_2 = \phi_1 - \pi/2$,

$$\lambda_i' = \frac{\lambda_a' \sec^2(\phi_a - \phi_j) - \lambda_b' \sec^2(\phi_b - \phi_j)}{\tan^2(\phi_a - \phi_j) - \tan^2(\phi_b - \phi_j)} \quad (22)$$

for $i = 1, 2$ and $j = 2, 1$. These two equations give expressions for unknown parameters of the principal directions (subscripts 1 and 2) in terms of known parameters of the lines subscripted a, b and c. In practice, given three independent longitudinal strain markers, equation (21) is solved first and the resultant value of ϕ_1 is substituted in equation (22) to obtain the principal reciprocal quadratic stretches λ_1' and λ_2'. A BASIC computer-program to perform these computations is appended, but a hand calculator would suffice. (For student tuition R. Burger, personal communication, advocates use of an Excel™ spreadsheet.) Note that the solution is not always real because all possible combinations of three radii of a quadratic do not necessarily lie on an ellipse (some lie on a hyperbola or, in special cases, on a straight line). When real data are employed, complex solutions either invalidate the assumption of homogeneous strain or reflect on the quality of the longitudinal strain markers. Also note that large errors ensue if two of the chosen lines are close to each other, especially if they are also close to the strain ellipse short axis.

EXAMPLE

An application of the above equations is illustrated using the problem in Ramsay & Huber (1983, p. 104). This problem was also chosen by Ragan (1987) to illustrate his matrix algebra method. Three longitudinal strain measurements are made on boudinaged belemnites of known orientation in the deformed state. Optimal result would be obtained by applying Ferguson's (1981) strain reversal technique to the three markers; however, for ease of comparison with the previously published results, the stretch and reciprocal quadratic stretch values from Ramsay & Huber are taken as given. Substituting $\lambda'_a = 0.41$ $(=1/1.56^2)$, $\lambda'_b = 0.30$ $(=1/1.82^2)$, $\lambda'_c = 0.18$ $(=1/2.38^2)$, $\phi_a = -68°$, $\phi_b = -220°$ and $\phi_c = 0°$ into equation (20) yields $\phi_1 = 4.32°$ (positive clockwise), which is close to Ramsay & Huber's graphically determined value of 4° and Ragan's (1987) matrix solution, 4.3° in the present reference frame. Substitution of ϕ_a, ϕ_b, ϕ_c and ϕ_1 into equation (22) yields $\lambda'_1 = 0.175$ $(=1/2.39^2)$ and $\lambda'_2 = 0.43$ $(=1/1.52^2)$, which compare with Ramsay & Huber's determinations, $\lambda'_1 = 0.18$ and $\lambda'_2 = 0.42$ and Ragan's of $\lambda'_1 = 0.18$ and $\lambda'_2 = 0.43$. While the second decimal place is not geologically significant, it is essential to carry at least two decimals through the calculation to avoid accumulating rounding errors.

APPLICATIONS

An immediate application will be found in the field of finite-element analysis. If the lines subscripted *a, b, c* are taken to be the boundaries of an element undergoing finite deformation, then the strain state within the element may be interpolated using equations (21) and (22). By virtue of the continuity of elements in a grid, neighboring elements will always yield compatible strain states. This simple solution to strain interpolation requires less data than the standard engineering approach using infinitesimal approximation parameters e and $\gamma/2$, because the displacement gradients are not required to be known. Only longitudinal strains, not rotations of the boundaries, are used and consequently the stretch, but not the rotational, component of deformation is interpolated.

In addition to the solution of two-dimensional problems, these equations have important implications for three-dimensional strain studies. While the amount of calculation involved is time-consuming without the aid of a computer, the method is far easier to understand than any alternatives and so is more likely to be of use to practical structural geologists. Given three strain ellipses measured in three arbitrary sections (not necessarily principal or even mutually orthogonal), the strain ellipse in any fourth section may be determined by a combination of standard stereonet procedure to find the stretches along the three lines of intersection of the fourth plane with the three measurement planes and the equations of this paper to deduce the fourth sectional strain ellipse. By repeated application of this procedure to a set of planes, say a set containing the vertical, the maximum and minimum semiaxes of the triaxial strain ellipsoid may be found. The intermediate ellipsoid axis is easily found by plotting the pole to the minimaxal plane and then solving for the strain ellipse in the vertical plane containing that pole (De Paor, in preparation).

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APPENDIX

The following program solves equations (21) and (22), and outputs the principal direction and principal stretches given three strain markers. The code is written for the ZBASIC™ compiler and runs on any Macintosh microcomputer; it should be easily translatable to suit other hardware and software. Do not type a carriage return between single-spaced lines; all distinct lines of code are double-spaced.

```
'Program Solve3stretches: ZBasic™ Source Code'

CLEAR

DEFDBL a–z:' Double precision'

DIM stretch(3),      lambdaprime(3),      phi(3)

DIM Pstretch(2),     Plambdaprime(2)      Pphi(2)

a = 1: b = 2: c = 3

pi = 4*ATN(1)

radians = pi/180

degrees = 180/pi


"Start"

FOR i = a TO c

        PRINT "Enter stretch and orientation of line ";CHR$(96+i);":"

        INPUT stretch(i), phi(i)

        lambdaprime(i) = 1/stretch(i)^2

        phi(i) = phi(i)*radians

NEXT i


'equation (21)'

numerator = 0: denominator = 0
```

```
FOR i = a TO c: j = i MOD c + 1: k = j MOD c + 1

        numerator = numerator + (lambdaprime(i)-
        lambdaprime(j))*COS(2*phi(k))

        denominator = denominator + (lambdaprime(i)-
        lambdaprime(j))*SIN(2*phi(k))

NEXT i

SELECT denominator

        CASE > 0

        Pphi(1) = -.5*ATN(numerator/denominator)

        CASE = 0

        Pphi(1) = -pi/4

        CASE < 0

        Pphi(1) = -.5*ATN(numerator/denominator)+pi

END SELECT

Pphi(2) = Pphi(1)-pi/2

'equation (22)'

FOR i = 1 TO 2: j = i MOD 2 + 1

        IF phi(a) = Pphi(i) OR phi(b) = Pphi(i) THEN
        phi(a) = phi(a)+.001:PRINT "/0 approximation"

        numerator = lambdaprime(a)/COS(phi(a)-Pphi(j))^2-
        lambdaprime(b)/COS(phi(b)-Pphi(j))^2

        denominator = TAN(phi(a)-Pphi(j))^2-
        TAN(phi(b)-Pphi(j))^2

        IF denominator <> 0 THEN Plambdaprime(i) =
        numerator/denominator ELSE "ComplexNumber"

        Pstretch(i) = 1/SQR(Plambdaprime(i))

NEXT i

IF Pstretch(1) = Pstretch(2) THEN "Isotropic"

LONG IF Pstretch(2) > Pstretch(1)

    SWAP Pstretch(2),Pstretch(1)

    SWAP Plambdaprime(2),Plambdaprime(1)

    SWAP Pphi(2),Pphi(1)

END IF

FOR i = 1 TO 2

        PRINT:PRINT "Principal direction: "; Pphi(i)*degree

        PRINT "Stretch, reciprocal quadratic stretch =
        ";Pstretch(i),Plambdaprime(i)

NEXT i

PRINT:PRINT "Strain Ratio = "; Pstretch(1)/Pstretch(2)

GOTO "Response"

"Isotropic"

PRINT "Strain ellipse is a circle of radius ";Pstretch(1);".
No unique axes."

GOTO "Response"

"Complex Number"

PRINT "No real solution for this data, O.K.?"

"Response"

INPUT "Type 'Q' to quit or any other key for another
calculation..."; anykey$

anykey$ = UCASE$(anykey$): IF anykey$ <> "Q" THEN
"Start" ELSE END
```